A Two-Location Problem

- Design a network connecting two locations, Anagon and Bregen, 200km apart
- Anagon: 5 employees, Bregen: 10 employees
- Each employee
  - call other site 4 times/day, avg. 5 minutes each
    \[4 \times 5 \times 15 = 300 \text{ min/day}\]
  - call others in the same office 10 times/day about joint work, each last avg. 3 minutes
    \[10 \times 3 \times 15 = 450 \text{ min/day}\]
  - Note here we are not using \(\binom{10}{2} + \binom{5}{2}\) for the # of calls
- How can we best provide the communications between the 2 cities?
Cost of Network Services and Components

- Communication services & components
  - PSTN (Public Switched Telephone Network), each line $25/m
  - Leased and small customer-owned phone switches:
    - PBX (Private Branch eXchange)
    - Network equipment purchase is typically amortized at 3% per month.
    - PBX would cost $60/month ($2000 * 3%)

**Question:** Can we use PBXs to trade node cost for line cost and reduce the total cost of the network?

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Public Switched Telephone Network (PSTN) Solution

PSTN, the straightforward solution:
order a PSTN line for each employee

PSTN carrier lines only
No PBXs!
The Cost of PSTN Solution

<table>
<thead>
<tr>
<th>Item</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 line charges</td>
<td>$375.00/month</td>
</tr>
<tr>
<td>Local calling</td>
<td>$487.50/month</td>
</tr>
<tr>
<td>Long distance calls</td>
<td>$2600.00/month</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>$3462.50/month</strong></td>
</tr>
</tbody>
</table>

- Assume an average number of work days/month is 21 2/3
  - Local: 150 local calls/day = 450 minutes/day = $487.5 / month
    \[450 \text{ min/day} \times $0.05/\text{m} \times \frac{65}{3} \text{ day} = $487.5/\text{month}\]
  - Long distance: 300 minutes /day = $2600 / month
    \[300 \text{ min/day} \times $0.4/\text{m} \times \frac{65}{3} \text{ day} = $2600/\text{month}\]

Something wrong? – low utilization of PSTN lines?

Utilization Analysis of PSTN Solution

- Local scenarios:
  - Each employee 30 min/day on outgoing local calls
    - no conference call, only point-to-point calls
    - 1 hour/day/employee utilization (outgoing + incoming, in average)

- Long-distance scenarios:
  - 5 employees at Anagon place 4 * 5 min * 5 = 100 min calls/day to Bregen
  - 10 employee at Bregen place 4 * 5 min * 10 = 200 min calls/day to Anagon
  - 300 min long distance calls are shared
    - among 5 employee at Anagon
      - that is, \(300\text{min} / 5 = 1\) hour/employee/day on long distance
    - among 10 employees at Bregen
      - that is, \(300\text{min} / 10 = 0.5\) hour/employee/day on long distance

- For a 8-hour day model
  - each phone at Anagon is busy \(\frac{2 \text{(1 local + 1 long)}}{8} = 25\%\) of the time
  - each phone at Bregen is busy \(\frac{121.5 \text{(1 local + 1 long)}}{8} = 25\%\) of the time

\[\longrightarrow\] Low Utilization of PSTN lines
**Design Principle 2.1**

*Good network designs tend to have many well-utilized components.*

- How about using a PBX at each location for handling all local calls?

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**PBX Solution**

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A: $487.5/month for local calls saved with $60*2=$120 amortized PBX cost, we actually save $367.5/month

B: reducing the Trunks at Bregen, since only 5 intersite simultaneous calls possible, $25*5=$125/month

Reliability degraded? ! (failure of either PBX)
Performance? Ok (if no delay at PBX)
Famous 2 camel-hump Telephone Traffic Daily Pattern

- Two call arrival peaks
- It is the busy hour that determines the Grade of Service (GoS) – blocking probability

**Erlang: the Traffic Measure Unit**

**Definition 2.1** If call arrive rate=$\lambda$ and departure rate=$\mu$, then the call intensity is $E = \frac{\lambda}{\mu}$ Erlangs.

- In honor of Danish Telephone engineer Erlang.

- Example 1. Calls arrive 2 per minute in average, and hold for an average of 3 min. Then, $\lambda=2$ and $\frac{1}{\mu}=3$, $E = \lambda/\mu = 6$ Erlangs.
  *Note that average hold time ($H$) = $1 / \text{departure rate}; H = 1/\mu$.

- Apparently one line cannot handle this amount of traffic. When a call comes and all lines are busy, the call is blocked.

  How many lines can reduce the blocking probability to x%?
The Erlang Calculation

° In our 2-location case, 15 employees place 4 long distance calls/day, each call lasts average 5 minute, assuming a 8-hour day model.

° What is the call (traffic) intensity for the day?
\[ \lambda = 15 \times 4 \text{ calls / 8 hr} = 15 / 2 \text{ calls/hr} \]
\[ H = 5 \text{ min/call} = 1/12 \text{ hr/call} \]
\[ \mu = 1 / H = 12 \text{ calls/hr} = 0.2 \text{ calls/min} \]
\[ E = \lambda / \mu = (15/2)/12 = 15/24 = 5/8 \text{ Erlangs}. \]

° Assuming 20% of the traffic in the busy hour, what is the call intensity in the busy hour?
\[ \lambda = 60 \text{ calls/day} \times 0.2 = 12 \text{ calls/hr} = 0.2 \text{ calls/min} \]
\[ \mu = 1 / H = 12 \text{ calls/hr} = 0.2 \text{ calls/min} \]
\[ E = \lambda / \mu = 12/12 = 1 \text{ Erlang}. \]

Queueing Theory for System with Loss

° Assume a telephone system with multiple lines.
  • When a call comes and all lines are busy, the call is blocked.
    Unlike data network, calls are not buffered or queued if lines are not available.
  • Or, you can consider it is a finite queue, i.e., after queue full (line busy), further call are blocked.

How many lines can reduce the blocking probability to x% for a given traffic density?

° Queueing theory can be used analyzed the telephone system’s performance, specifically the blocking probability.
  • Loss network – telephone system, no queue (or full queue)
  • Data network, queueing-delay, no loss (infinite queue)
Calculating the Blocking Probability

- Define $P_i$ the state probabilities of time $t$, i.e., when a call comes at time $t$, it is the probabilities that the system is at state $S_i$ ($i$ phones are in progress). $P_i \geq 0, \sum_{i=0}^{m} P_i = 1$.
  - $P_m$ is the blocking probability in a system with $m$ lines.

- In the 2-location case, within the busy hour, arrival rate $A$ ($\lambda$) = 0.2 calls/min, departure rate $D$ ($\mu$) of one line = 0.2 calls/min.
  - When 1 call in progress, departure rate is 0.2 calls/min
  - When 2 calls in progress, departure rate is 0.4 calls/min
  - When $m$ calls are in progress, departure rate is $m \times 0.2$ calls/min

- $P_0$ = $P_1$
- $P_2 = \frac{1}{2} P_1$
- $P_3 = \frac{1}{3} P_2$
- $P_4 = \frac{1}{4} P_3$
- $P_5 = \frac{1}{5} P_4$
- $P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 1$
- $P_0 = \frac{120}{326}$ ……
- $P_5 = \frac{1}{326}$ \( \Rightarrow \) the blocking probability is $1/326$. 

0.2 * $P_0 = 0.2 * P_1$ if system is at equilibrium (Heat equation in Physics!)
Loss with \( m \) Lines (\( m \) servers, no queue)

- \( P_k = \frac{E^k}{k!} P_0 \)
- \( \sum_{k=0}^{m} P_k = \sum_{k=0}^{m} \frac{E^k}{k!} P_0 = 1 \to P_0 = \sum_{k=0}^{m} \frac{E^k}{k!} \)
- Probability of loss \( = P_m = \frac{E^m}{m!} P_0 = \frac{1}{\sum_{k=0}^{m} \frac{E^k}{k!}} = B(E, m) \)
- \( B(E, k) = \frac{E B(E, k-1)}{EB(E, k-1) + k} \quad \forall k = 1, 2, \ldots, m \)

\( B(E, 0) = 1 \)

Erlang-B Function

- \( B(E, m) \) is called Erlang-B function and denote the probability of blocking when \( E \) Erlangs of traffic is offered to \( m \) channels.
- Here \( E \) is the offered load;
  - \( E B(E, m) \) Erlangs is the lost (traffic);
  - \( E (1-B(E, m)) \) Erlangs are carried (traffic)

\( \text{Offered} \quad E \quad \text{Carried} \quad E (1-B) \quad \text{EB} \quad \text{Lost} \)

- \( \text{(Per)Line Utilization} = E (1-B(E, m))/m \)

Average Utilization
**Design Intersite Links**

Q1: Given we can tolerate x% blocking probability, how many lines we actually need?

- Here \(E=1\) (busy hour)
- Blocked = \(B(E,m)\)
- Carried load = \(E(1-B(E,m))\)

If \(x = 1\) and \(2\), how many lines are needed, respectively?

<table>
<thead>
<tr>
<th>Total links</th>
<th>Blocking</th>
<th>Carried</th>
<th>(q_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1/2 = 0.500)</td>
<td>0.500</td>
<td>0.500</td>
</tr>
<tr>
<td>2</td>
<td>(1/5 = 0.200)</td>
<td>0.800</td>
<td>0.300</td>
</tr>
<tr>
<td>3</td>
<td>(1/16 = 0.0625)</td>
<td>0.9375</td>
<td>0.1325</td>
</tr>
<tr>
<td>4</td>
<td>(1/65 = 0.015)</td>
<td>0.985</td>
<td>0.0475</td>
</tr>
<tr>
<td>5</td>
<td>(1/326 = 0.003)</td>
<td>0.997</td>
<td>0.012</td>
</tr>
</tbody>
</table>

**Simplified Traffic Profile**

- Instead of the 2 camel hump traffic pattern, simplify it to two levels: peak and off-peak.

<table>
<thead>
<tr>
<th>Number of hours</th>
<th>Usage/hour</th>
<th>Dial cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>60 minutes/hour</td>
<td>$1040</td>
</tr>
<tr>
<td>6</td>
<td>30 minutes/hour</td>
<td>$1560</td>
</tr>
</tbody>
</table>

- Dial Cost: 60 min/hour * 2 hour * $0.4/min * (21+2/3)day/mon = 1040/mon
- 30 min/hour * 6 hour * $0.4/min * (21+2/3)day/mon = 1560/mon
Reduce Cost by Using Leased Lines

Q2: Given leased lines (fixed $275 monthly fee) and PSTN dial lines ($0.4/min), how many leased lines should be used in the design?

- The leased line costs $275/month, which can replace two PSTN lines, one on each end, for a total of $50
  - Additional cost: $275 - $50 = $225

- Strategy: Instead of charged by # of calls, place calls on leased lines first (pay monthly), the calls placed on the leased line save $0.4/min

- Order the 5 intersite links, calculate the traffic value carried by each link, then determine if the link should be leased.

Let us figure out the cost saving of each leased line

- First focus on busy-hour analysis
- Then, focus on off-peak analysis

Busy Hours Analysis for Leased Line Saving

What is the value of busy-hours traffic carried by a single leased line?

- With 60min/hour usage, traffic is E = 60/60 = 1 Erlang.

From the table above, the fraction of calls loaded on the Link 1 ($q_1$) is 0.5. The other half calls got blocked.

- The value of carried dialup traffic: 0.5 * $1040/month = $520/month
- The net equipment cost of a leased line = $275 - $50 (replaced a PSTN line at each site) = $225/month
- The net cost saving by leasing: $520 - $225 = $295/month
- It is justifiable that the first link should be leased!
2nd Leased Line Saving in Busy Hours

- In busy hours, the 2nd link will carry 0.3 traffic
  - The value of the carried dialup traffic: \(0.3 \times 1040 = 312\) month
  - \$312 > \$225 (by leasing) \(\rightarrow\) 2nd leased line is also justified
- The third link only carries 0.1325 busy hour traffic
  - The value of the carried dialup traffic: \(0.1325 \times 1040 = 143\) month
  - \$143 < \$225 \(\rightarrow\) 3rd leased line is NOT justified
- We need to see if the cost saving of off-peak hour usage of the 3rd leased line adds up to \$225! If so, still justifiable!

<table>
<thead>
<tr>
<th>Traffic Value</th>
<th>$520</th>
<th>$312</th>
<th>$143</th>
<th>$49</th>
<th>$12</th>
</tr>
</thead>
</table>

Off-Peak Analysis of 3rd Leased Line

- Off-Peak hours traffic: 30/60 = 0.5 Erlang.
- Blocking probability of Link 1: \(B(0.5, 1) = 0.5/(0.5 + 1) = 1/3 = 0.333\)
- The fraction of calls carried by Link: \(1 – 0.333 = 0.667\)
- The traffic value carried by Link 1: \(0.667 \times 1560 = 1040\) month
- The fraction of calls carried by Link 3:
  \[
  (1 - B(0.5, 3)) - (1 - B(0.5, 2)) = B(0.5, 2) - B(0.5, 3) = (1/13) - (0.5*B(0.5,2))/(0.5*B(0.5,2)+1) = 1/13 - 1/79 = 0.0643
  \]
- The traffic value carried by Link 1: \(0.0643 \times 1560 = 100.25\) month
- \$100.25 (off-peak) + \$143 (peak) = \$243.25 > \$225 \(\rightarrow\) justified!
## Final Design

### Link Details

<table>
<thead>
<tr>
<th>Link #</th>
<th>Type</th>
<th>Cost</th>
<th>Value of carried traffic (peak + off-peak)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>leased</td>
<td>$275</td>
<td>$1,560</td>
</tr>
<tr>
<td>2</td>
<td>leased</td>
<td>$275</td>
<td>$712</td>
</tr>
<tr>
<td>3</td>
<td>leased</td>
<td>$275</td>
<td>$243.25</td>
</tr>
<tr>
<td>4-5</td>
<td>PSTN</td>
<td>$184.75</td>
<td>$84.75 (2600-1560-712-243.25)</td>
</tr>
</tbody>
</table>

## Cost of Final Design

- **2 PBXs**
- **3 leased lines**
- **4 PSTN lines**
- **Dialup cost**

### PSTN Cost

$3462.5 - $1129.75 = $2332.75/month saving